3 Sobolev Spaces

Exercise 3.1. Let $\varphi \in \mathscr{D}(I)$. We have

$$\begin{aligned} \langle u', \varphi \rangle = -\langle u, \varphi' \rangle &= -\int_0^1 u(0)\varphi'(x)\,dx - \int_0^1 \left(\int_0^x v(t)\,dt\right)\varphi'(x)\,dx \\ &= -\int_0^1 \int_0^x v(t)\varphi'(x)\,dtdx = -\int_0^1 \int_t^1 v(t)\varphi'(x)\,dxdt \\ &= -\int_0^1 v(t)\left(\int_t^1 \varphi'(x)\,dx\right)dt = \int_0^1 v(t)\varphi(t)\,dt = \langle v, \varphi \rangle. \end{aligned}$$

Exercise 3.2. We consider first the case $u = v = 0$. We prove that if $(0, f') \in W^{1,1}(I)$ then $f' = 0$. In particular, since $(0, v'), (0, u') \in W^{1,1}(I)$ we will deduce $u' = 0 = v'$. Since $f' \in L^1(I)$ it is sufficient to prove that

$$\int_I f'\varphi = 0 \qquad \forall \varphi \in C_c^\infty(I).$$

Since $(0, f') \in W^{1,1}(I)$, then there exists a sequence $(f_h)_h \in C^1(I)$ such that

$$f_h \to 0 \quad \text{in } L^1(I), \qquad \text{and} \qquad f_h' \to f' \quad \text{in } L^1(I).$$

In particular

$$\int_I f_h'\varphi = [f_h\varphi]_0^1 - \int_I f_h\varphi' = -\int_I f_h\varphi'.$$

Since $\varphi, \varphi' \in L^\infty(I)$, then

$$\lim_{h\to\infty} \int_I f_h'\varphi = \int_I f'\varphi, \qquad \text{and} \qquad \lim_{h\to\infty} \int_I f_h\varphi' = 0$$

and this proves our claim.

Let us now consider the general case $u = v$: set $z := u - v = 0$. Then by the previous case $z' = u' - v' = 0$, namely $u' = v'$.

Exercise 3.3. From Exercise 1. we know that if $u \in W^{1,1}(I)$, then $u(x) = u(y) + \int_y^x v(t)\,dt$, where $v \in L^1(I)$ is the distributional derivative of u. For this reason let $u = H(t - \frac{1}{2})$, then its distributional derivative $u'(x) = \delta(x - \frac{1}{2}) \notin L^1(I)$, thus in general, step functions do not belong to $W^{1,1}(I)$.

Let $u(x) = x^\alpha$. Since $u \in C^1(I)$, we have that $u'(x) = \alpha x^{\alpha-1}$, from which we conclude that $u' \in L^1(I)$ if and only if $\alpha > 0$.

Exercise 3.4. Let $\{u_n\}_n \subset W_0^{1,1}(I)$ be a Cauchy sequence in the $W^{1,1}(I)$ norm. Since $W^{1,1}(I)$ is a Banach space, we get that $u_n \to u \in W^{1,1}(I)$. Moreover, since $u_n(x) = \int_0^x u'(t)\,dt$, we get that $\sup_{x\in I}|u_n(x)| \leq \|u_n'\|_{L^1(I)}$. In particular this shows that $\{u_n\}_n$ is also a Cauchy sequence in $C^0(I)$. Thus $u_n(x) \to u(x)$ for every $x \in I$, from which we conclude

$$u(0) = \lim_{n\to\infty} u_n(0) = \lim_{n\to\infty} u_n(1) = u(1) = 0.$$

Let now $u \in W_0^{1,1}(I)$. Then $u(x) = \int_0^x u'(t)\,dt$, thus

$$\|u\|_{L^1(I)} \leq \int_0^1 \int_0^x |u'(t)|\,dtdx = \int_0^1 \int_t^1 |u'(t)|\,dxdt = \int_0^1 (1-t)|u'(t)|\,dt \leq \|u'\|_{L^1(I)},$$

from which we conclude that

$$\|u'\|_{L^1(I)} \leq \|u\|_{L^1(I)} + \|u'\|_{L^1(I)} = \|u\|_{W^{1,1}(I)} \leq 2\|u'\|_{L^1(I)}.$$

Exercise 3.5. If $u \in W^{1,p}(I)$, then $u' \in L^p(I)$. Thus, by Hölder inequality we have that

$$|u(x)-u(y)| \leq \int_y^x |u'(t)|\,dt \leq \left|\int_y^x 1\,dt\right|^{1-\frac{1}{p}} \|u'\|_{L^p(I)} = \|u'\|_{L^p(I)}|x-y|^{1-\frac{1}{p}}.$$

To show that in general the opposite inclusion fails, take $u(x) = x^{1-\frac{1}{p}}$. Then $u \in C^{0,1-\frac{1}{p}}(I)$, but $u'(x) = \left(1-\frac{1}{p}\right)x^{-\frac{1}{p}} \notin L^p(I)$, thus u cannot belong to $W^{1,p}(I)$.